PROBABILISTIC ANALYSIS OF A SIMPLE MAC SCHEME FOR AD HOC WIRELESS NETWORKS (SHORT PAPER)

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ABSTRACT

The channel access scheme is critical for the performance and energy efficiency of ad hoc wireless networks. We present a rigorous analysis of a simple and fully scalable MAC technique and demonstrate how the trade-off between energy consumption and delay can be resolved. The analysis is based on a probabilistic model that encompasses all the relevant adversities of wireless networks. For Rayleigh fading channels, it turns out that the performance analysis can be carried out independently for effects caused by noise and effects caused by interference. We study the case of one-dimensional networks, and, as an upper performance bound we also provide the results for optimum scheduling.

1. INTRODUCTION

For performance assessment of multi-hop wireless networks, it is often assumed that the radius for a successful transmission of a packet has a fixed and deterministic value R, irrespective of the condition and realization of the wireless channel [1, 2]. Furthermore, the interference range is often taken to be identical to cR with some $c \geqslant 1$. Such simplified link models ignore the probabilistic nature of the wireless channel and the fact that the *signal-to-noise-and-interference ratio*, that determines the success of a transmission, is a random variable.

Fading, noise, and interference are the key adversities in a wireless network. In this work, we use a link model that is derived from the physical layer and takes those effects into account. Nevertheless, it is analytically tractable and thus yields valuable insight into the network behavior.

For the network, it is assumed that N nodes are arranged at positions x_i. Nodes are transmitting packets independently in every timeslot $k = 0, 1, 2, \ldots$ with equal probability p and power P_0. The packets are of equal length and fit into one time slot. Traffic is bidirectional, *i.e.*, the packet's destination can be any of the nearest neighbors. The results for our simple scheme provide a useful analytical lower bound; as an upper bound, the performance of the optimum MAC scheduler is determined.

The case of one-dimensional networks is given special consideration. Its practical significance lies in the fact that ideally, any multi-hop connection in a network is by itself a one-dimensional network.

The partial support of the DARPA/IXO-NEST Program (AF-F30602-01-2-0526) is gratefully acknowledged.

2. THE PROBABILISTIC LINK MODEL

We assume a narrowband multipath wireless channel with a coherence time longer than the packet transmission time. The channel can then be modeled as a flat Rayleigh fading channel [3] with AWGN z. Therefore the received signal is $y_k = a_k x_k + z_k$, where a_k is the path loss multiplied by the fading coefficient. The variance of the noise process is denoted by σ_z^2. With Rayleigh fading, the received signal power is exponentially distributed.

The transmission from node i to node j is successful if the signal-to-noise-and-interference ratio (SINR) γ_{ij} is above a certain threshold Θ that is determined by the communication hardware, and the modulation and coding scheme (normally between 1 and 100 (0dB-20dB):

$$\gamma_{ij} \geqslant \Theta \tag{1}$$

With the assumptions above, γ_{ij} is a discrete random process with exponential distribution: $p_{\gamma_{ij}}(x) = 1/\bar{\gamma}_{ij}\, e^{-x/\bar{\gamma}_{ij}}$ with mean

$$\bar{\gamma}_{ij} = \frac{\bar{P}_{ij}}{\sigma_z^2 + \sigma_{ij}^2}\,. \tag{2}$$

$\bar{P}_{ij}$ denotes the average received signal power over a distance $d_{ij} = \|x_i - x_j\|_2$: $\bar{P}_{ij} = P_0 d_{ij}^{-\alpha}$, where P_0 is proportional to the transmit power[1], and the path loss exponent is $2 \leqslant \alpha \leqslant 5$.

σ_{ij}^2 is the interference power affecting a transmission from i to j. It is the sum of all the undesired signals and can be expressed as

$$\sigma_{ij}^2 = \sum_{n \in \mathcal{T} \setminus \{i\}} \bar{P}_{nj}\,, \tag{3}$$

where $\mathcal{T}$ is the set of nodes that are transmitting in a given time slot. Note that node j itself can be a member of $\mathcal{T}$. If it is, every transmission from any node to node j is bound to fail. In [2, 4], the SINR is defined in a similar way. However, the transmission is considered to be successful whenever $\bar{\gamma}_{ij}$ is bigger than some threshold. Hence, only the large-scale path loss is considered, while the probabilistic nature of the fading channel is ignored.

It follows from (2) that the *reception probability* over a link ij, $p_{ij} = \mathbb{P}[\gamma_{ij} \geqslant \Theta]$, is

$$p_{ij} = e^{-\Theta/\bar{\gamma}_{ij}} = e^{-\frac{\Theta\sigma_z^2}{\bar{P}_{ij}}} \cdot e^{-\frac{\Theta\sigma_{ij}^2}{\bar{P}_{ij}}}\,. \tag{4}$$

[1] This equation does not hold for very small distances. So, a more accurate model would be $\bar{P}_{ij} = P_0' \cdot (d_{ij}/d_0)^{-\alpha}$, valid for $d_{ij} \geqslant d_0$, with P_0' as the average value at the reference point d_0, which should be in the far field of the transmit antenna. At 916MHz, for example, the near field may extend up to 3-4ft (several wavelengths).

Clearly, the reception probability is the product of two probabilities, where one factor is determined by the noise and the other one by the interference. Thus, instead of the SINR γ_{ij}, we can define the two independent exponentially distributed random variables, namely the SNR γ_{ij}^Z with mean $\bar{\gamma}_{ij}^Z = \bar{P}_{ij}/\sigma_z^2$ and the SIR γ_{ij}^I with mean $\bar{\gamma}_{ij}^I = \bar{P}_{ij}/\sigma_{ij}^2$, and formulate the condition for successful reception as two independent conditions that both need to be satisfied:

$$\gamma_{ij}^Z \geqslant \Theta \quad \text{and} \quad \gamma_{ij}^I \geqslant \Theta \;\Leftrightarrow\; p_{ij} = e^{-\Theta/\bar{\gamma}_{ij}^Z} \cdot e^{-\Theta/\bar{\gamma}_{ij}^I} \tag{5}$$

This allows an independent analysis of the effect caused by noise and the effect caused by interference. Whereas the first probability can be improved by increasing the transmit power, the interference probability is independent of P_0, for both $\bar{P}_{ij}$ and σ_{ij}^2 scale with P_0.

3. NETWORK PERFORMANCE ANALYSIS

Based on the model presented in the previous Section, we are analyzing the network performance in terms of throughput and delay as functions of the parameters p and P_0 given α, σ_z^2, and Θ. As shown in (4) and (5), the analyses with respect to noise and with respect to interference can be carried out independently.

3.1. Noise Analysis

We first determine the performance of a network without interference. In this case, only one node is transmitting, so we can consider a two-node network. The link success or reception probability over this link, p_r, is then given by

$$p_r := \mathbb{P}[\gamma \geqslant \Theta] = e^{-\frac{\Theta\sigma_z^2}{P_0\, d^{-\alpha}}} . \tag{6}$$

To achieve a link reliability of P_L, retransmissions are necessary if the reception probability p_r is smaller than P_L. The total number of transmissions is given by

$$n_t = \begin{cases} \frac{\log(1-P_L)}{\log(1-p_r)} & p_r < P_L \\ 1 & p_r \geqslant P_L \end{cases} . \tag{7}$$

For our analysis, we neglect the fact that n_t can only assume integer values. Focusing on the interesting range where $p_r < P_L$, the total energy consumption to achieve a reliability of P_L is

$$E_{P_L}(P_0) = n_t\, P_0 = P_0\, \frac{\log(1-P_L)}{\log\left(1 - e^{-\frac{\Theta\sigma_z^2}{P_0\, d^{-\alpha}}}\right)} . \tag{8}$$

The location of the minimum of $E_{P_L}(\cdot)$ does not depend on the desired P_L:

$$P_0^{\text{opt}} = \arg\min_{P_0} E_{P_L}(P_0) = \frac{\Theta\sigma_z^2 d^\alpha}{\ln 2} . \tag{9}$$

The factor $1/\ln 2 \approx 1.44$ corresponds to 1.6dB, hence the transmit power that minimizes the overall energy consumption is 1.6dB above $\Theta\sigma_z^2$. From (6), it can be derived that this point corresponds to a link success probability of $p_r = 0.5$. The minimum energy value is then given by

$$E_{P_L}(P_0^{\text{opt}}) = \Theta\sigma_z^2 d^\alpha\, \frac{-\ln(1-P_L)}{(\ln 2)^2} . \tag{10}$$

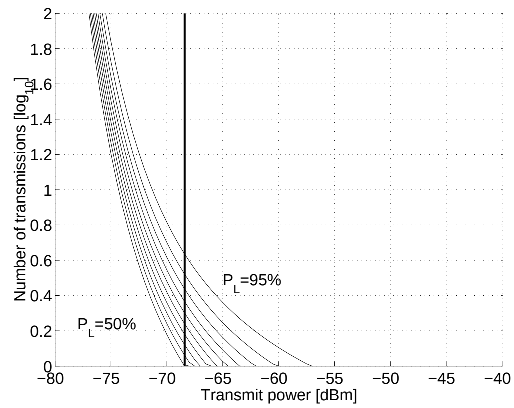

Figure 1: Total number of transmissions to achieve a link reliability of P_L (logarithmic scale).

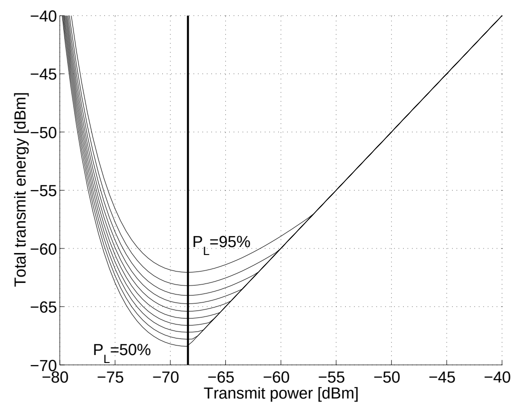

Figure 2: Total energy consumption as a function of P_0 for $P_L = 0.50, 0.55, \ldots, 0.95$. The vertical line indicates the optimum in terms of energy.

Figs. 1 and 2 show the dependence of the number of transmissions and the total energy consumption as a function of P_L. The energy is expressed in dBm, which corresponds to dB(mJ) if the duration of a packet transmission were 1s.

The *delay* that a data packet experiences is proportional to n_t. Comparing the minimum required transmit energy $E_{P_L}(P_0^{\text{opt}})$ with the single-transmission ($n_t = 1$) energy $E_{P_L}(P_0^{\text{single}})$

$$E_{P_L}(P_0^{\text{single}}) = d^\alpha\, \frac{\Theta\sigma_z^2}{-\ln P_L} \tag{11}$$

for a desired link reliability P_L, we get a ratio of

$$\frac{E_{P_L}(P_0^{\text{opt}})}{E_{P_L}(P_0^{\text{single}})} = \frac{\ln(1-P_L)\;\ln P_L}{(\ln 2)^2} , \tag{12}$$

which is 1 for $P_L = 0.5$ and achieves -10dB at $P_L = 0.99$.

3.2. Interference Analysis

In this Section, we consider a network of N nodes where the reception is only corrupted by interference, not by noise. The performance of the network is the number of successful packet transmissions in a given time slot, denoted by $g_N(p)$.

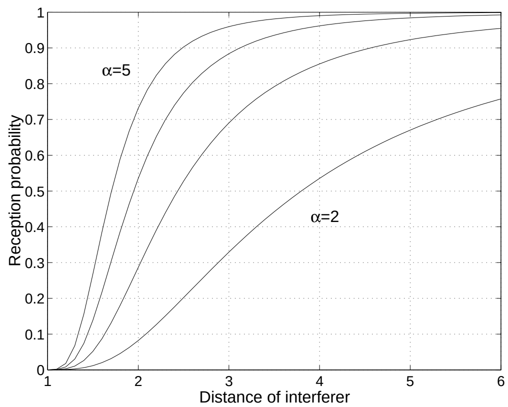

Figure 3: Reception probability as a function of the interferer distance for $\Theta = 10$ and $\alpha = 2, 3, 4, 5$.

In general, for a transmission over unit distance with one interferer at distance d, the mean SIR is $\bar{\gamma}^I = d^\alpha$, yielding a reception probability of (see Fig. 3)

$$p_r = e^{-\Theta d^{-\alpha}} \,. \tag{13}$$

For multiple interferers, the respective probabilities p_{r_k} are to be multiplied:

$$p_r = \prod_{k=1}^{n} e^{-\Theta d_k^{-\alpha}} = e^{-\Theta \sum_{k=1}^{n} d_k^{-\alpha}} \,. \tag{14}$$

Small networks.
Small networks can be rigorously analyzed. For $N = 2$, the transmission is successful if one node transmits and the other one does not, *i.e.*, $g_2(p) = 2p(1-p)$ with a maximum of $1/2$ at $p = 1/2$. For 3 nodes, either one node may transmit (no interference) or two nodes transmit, causing mutual interference. If the nodes are on a line with unit distance and one of the end nodes is the receiver, the transmission is successful with probability $e^{-\Theta\, 2^{-\alpha}}$. Since there are two such cases, but in each, the center node only transmits to the non-transmitting node with probability $1/2$, the performance is $g_3 = 3p(1-p)^2 + e^{-\Theta\, 2^{-\alpha}} p^2(1-p)$. If there is an interferer at the same distance as the intended transmitter, the reception probability is $e^{-\Theta}$.
In general, the number of correctly received packets $g_N(p)$ in a network with N nodes as a function of the transmit probability p can be expressed as

$$g_N(p) = \sum_{k=1}^{N-1} c_k \, p^k \, (1-p)^{N-k} \,, \tag{15}$$

since in every timeslot, k nodes transmit and $N - k$ do not. The cases where 0 nodes or N nodes transmit do not contribute to $g_N(p)$. $g_N(p)$ is an N-th order polynomial with $g_N(0) = g_N(1) = 0$ and a single maximum in $(0, 1/2]$. At $p = 0$, its derivative is $c_1 = N$, indicating that for small p, the number of successful transmissions is proportional to p. The other coefficients c_k depend on Θ, α, and the topology of the network.

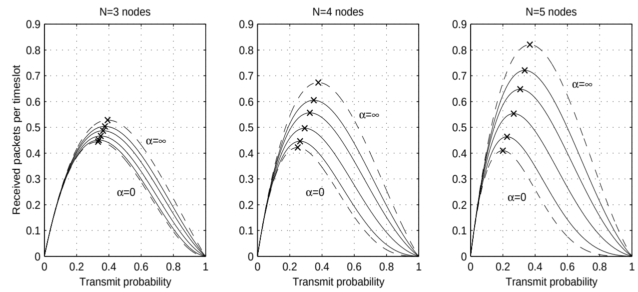

Figure 4: Received packets per timeslot as a function of the transmit probability for a one-dimensional network for $\alpha = 2, 3, 4, 5$ (solid) and $\alpha = 0, \infty$ (dashed). $\Theta = 10$.

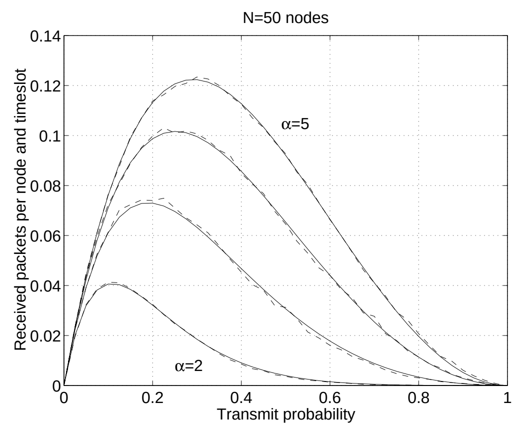

Figure 5: Received packets per timeslot and node for $\Theta = 10$ and $\alpha = 2, 3, 4, 5$ for a one-dimensional network with $N = 50$ nodes. The solid line is the analytical approximation, the dotted line the simulation result.

One-dimensional networks with unit node distance.
In this case, the node distances $d(j)$ in (15) are integers, and the number of terms in c_k is substantially reduced.
For the case where every node randomly chooses its left or right neighbor as its destination, Figure 4 depicts the functions $g_N(\cdot)$ for $N = 3, 4, 5$ and $\alpha = 2, 3, 4, 5$ with $\Theta = 10$. As lower and upper bounds, the cases $\alpha = 0$ and $\alpha = \infty$ are also included.
For larger N, the manual derivation of the coefficients c_k is still theoretically feasible, but not practical. Since we know that the exponential term in (14) must be present in the approximation, we can deduce that

$$\hat{g}_N(p) = Np(1-p)^2 e^{-qp} \,, \qquad q > 0 \,, \tag{16}$$

is a good analytical approximation. q is a function of α and Θ. The factor $p(1-p)^2$ arises because for a successful transmission, there must be a group of three adjacent nodes where only the middle one transmits. The exponential term accounts for the interference. The parameter q can be determined by a numerical least squares optimization. In Fig. 5, both a simulation and the corresponding analytical function are plotted. Indeed, they are almost perfectly identical. As expected, for large N, the performance per node converges to a constant.

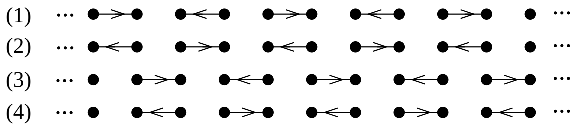

Figure 6: The four phases of the optimum transmit schedule for $q = 2$ with bidirectional traffic.

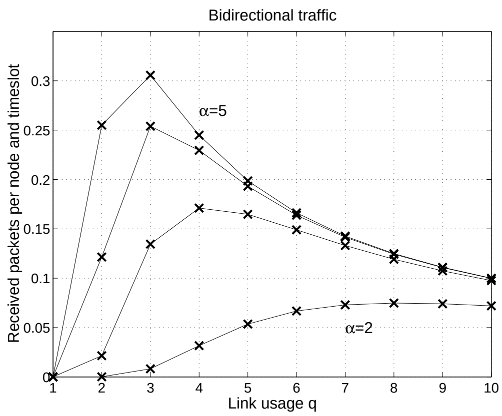

Figure 7: Received packets per timeslot and node for $\Theta = 10$ and $\alpha = 2, 3, 4, 5$ for a large one-dimensional network where every q-th link is used in every timeslot.

Comparison with the optimum scheduler.
With the proposed link model, the performance of the optimum scheduler can be determined in a straightforward manner. Assume that in a one-dimensional network, every q-th link is used in a given timeslot. Fig. 6 shows the best scheduling scheme for $q = 2$ for bidirectional traffic ($2q$ phases). In Fig. 7, the throughput as a function of q is plotted. Optimum scheduling is achieved at $q = 8, 4, 3, 3$ for $\alpha = 2, 3, 4, 5$. Note the similarity of the curve with Fig. 5. The throughput ratio between the simple MAC scheme and the optimum one is in $[0.4, 0.5]$ for $\alpha = 5, 4, 3, 2$. For smaller Θ, the lower bound of the ratio approximates $1/e$, which corresponds to the ratio of slotted ALOHA vs. optimum scheduling for the multiple access channel.
Interestingly, the bidirectional use of a connection does not cut the throughput in half. The unidirectional case, where every q-th link is used in the same direction, is shown in Fig. 8. Clearly, the maximum per-node throughput is lower, and, for the same q, the average distance between the receiver and the interfering transmitters is lower. Maximum throughput is achieved at $q = 8, 5, 4, 3$ for $\alpha = 2, 3, 4, 5$. The "bidirectional gain" ranges from 1.4% ($\alpha = 2$) to 27% ($\alpha = 5$). For $\Theta = 0$dB, it increases to 10%–45%.

4. CONCLUSIONS

With the proposed probabilistic link model, the transmit power P_0 and the transmit probability p for the simple MAC scheme can be independently optimized. The resulting performance is simply the product of the reception probability (from the noise analysis) and the performance $g_N(\cdot)$ (from the interference analysis). Hence, increasing P_0 does *not* have an adverse effect on the packet transmission.

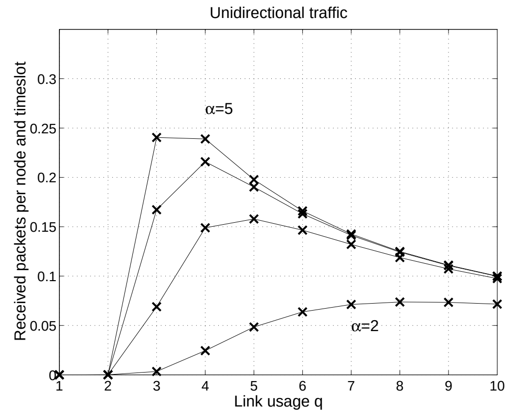

Figure 8: Received packets per timeslot and node for $\Theta = 10$ and $\alpha = 2, 3, 4, 5$ for a large one-dimensional network where every q-th link is used unidirectionally.

Noise analysis. The useful power range is $P_0 \in [\Theta\sigma_z^2, \Theta\sigma_z^2 + 20]$ (in dBm). In this range, the trade-off between delay (number of transmissions to achieve a desired reliability) and power consumption can be analytically resolved. The optimum transmit power with respect to energy consumption is only 1.6dB above $\Theta\sigma_z^2$, which corresponds to a reception probability of 50%.
Interference analysis. The performance polynomial $g_N(p)$ depends on Θ and, almost linearly, on the path loss exponent α. As a rule of thumb, we can say that in a one-dimensional network with unit distance, with $\Theta = 10$ and high α (and high transmit power), the per-node performance is approximately 0.1, *i.e.*, a node can successfully transmit in at most 10% of the timeslots. The transmit probability should be between 25% and 30%. In terms of energy, p should be kept as low as possible. For small p ($p < p_{\max}/2$), the performance is proportional to p. This is the energy-neutral area, since an increase in p results in a proportional increase in the performance. For larger p, the delay is reduced, at the expense of higher energy consumption.
Compared with the optimum scheduler, where the per-node throughput is about 0.25, the loss in throughput is approx. $1/e$; the performance of every practical MAC scheme lies in between. The gain from using links bidirectionally is about 25% for $\alpha > 3$ and $\Theta \lessapprox 10$dB.

5. REFERENCES

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